8.4 - Matrix Exponential

Recall that $e^x = \sum_{k=1}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$ This is the power series representation for the exponential function. In the same way, we can define a matrix exponential.

Definition: For any $n \times n$ matrix $\mathbf{A}$,

$A^2 = AA, \quad A^3 = \underbrace{AA}_{A^2}A = A\overbrace{AA}^{A^2}$

$$e^{\mathbf{A}t} = \underbrace{\mathbf{I}}_{n \times n \text{ identity}} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^k\frac{t^k}{k!} + \cdots = \sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^k}{k!}$$

Example: Compute $e^{\mathbf{A}t}$ and $e^{-\mathbf{A}t}$.

$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$e^{At} = I + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 \frac{t^2}{2} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^3 \frac{t^3}{3!} + \cdots$$

$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}A^2 = A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A^4 = A^2A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

So $e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}t^2 & 0 \\ 0 & \frac{1}{2}t^2 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{3!}t^3 \\ \frac{1}{3!}t^3 & 0 \end{pmatrix} + \cdots$

$\sum_{i=1}^{\infty} \frac{1}{(2i)!}t^{2i}$

$$e^{At} = \begin{pmatrix} 1 + \frac{1}{2}t^2 + \frac{1}{4!}t^4 + \cdots & t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \cdots \\ t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \cdots & 1 + \frac{1}{2}t^2 + \frac{1}{4!}t^4 + \cdots \end{pmatrix}$$

$2i+1$

6.1:
$$e^{At} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

In general, $e^{A+B} \neq e^{A}e^{B}$. But if $AB = BA$, then $e^{A+B} = e^{A}e^{B}$.

Note that since A commutes with itself, $(AA = AA)$, $e^{At}e^{-At} = e^{At-At}$

$$= e^{0 \text{ matrix}} = I$$

so $(e^{At})^{-1} = e^{-At}$

$$e^{-At} = \begin{pmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix}$$

Consider $\mathbf{X}' = \mathbf{A}\mathbf{X}$. If $\mathbf{X} = e^{\mathbf{A}t}\mathbf{C}$, then $\mathbf{X}' = \frac{d}{dt}\left(e^{\mathbf{A}t}\mathbf{C}\right)$. To find $\frac{d}{dt}\left(e^{\mathbf{A}t}\right)$, we use the definition:

$$\frac{d}{dt}\left(e^{\mathbf{A}t}\right) = \frac{d}{dt}\left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \mathbf{A}^3\frac{t^3}{3!} + \cdots\right) = A + A^2 t + A^3\frac{t^2}{2} + A^4\frac{t^3}{3!} + \cdots$$

$$= A\left(I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{3!} + \cdots\right) = A e^{At}$$

thus, $\frac{d}{dt}(e^{At}) = Ae^{At}$

$\mathbf{X}' = \mathbf{A}e^{\mathbf{A}t}\mathbf{C}$

So for $\mathbf{X}' = \mathbf{AX}$, with $\mathbf{A}$ containing constant entries, $\mathbf{X} = e^{\mathbf{A}t}\mathbf{C}$ is a solution. (Note: $\mathbf{C}$ is a column matrix of arbitrary coefficients.)

Example: Use the matrix exponential to find the general solution of the given system.

$\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{X}$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow e^{At} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$

So $\boxed{\vec{X} = \underbrace{\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}}_{\Phi} \vec{C}}$ $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ arbitrary constants

This works well when A^k is nice - either has patterns or if $A^k = 0$ for some k.

Nonhomogeneous {
Initial-value problems

$e^{\mathbf{A}t} = \mathbf{\Phi}$ is the fundamental matrix for the system, so variation of parameters yields, the general solution

$$\mathbf{X} = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)\,ds$$

Note: $e^{-\mathbf{A}s}$ is $e^{\mathbf{A}t}$ with t replaced by $-s$. In our work, we will take $t_0 = 0$.

Example: Find the general solution of the given system.

$$X' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} X + \begin{pmatrix} 3 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1^2 & 0 \\ 0 & 2^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1^3 & 0 \\ 0 & 2^3 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} t + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^2 \frac{t^2}{2} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^3 \frac{t^3}{3!} + \cdots$$

Note: $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^k = \begin{pmatrix} 1^k & 0 \\ 0 & 2^k \end{pmatrix}$

$$e^{At} = \begin{pmatrix} 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots & 0 \\ 0 & \underbrace{1 + 2t + \frac{2^2 t^2}{2} + \frac{2^3 t^3}{3!} + \cdots}_{e^{2t}} \end{pmatrix}$$

For $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$

$$\vec{X}_c = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \vec{C}$$

$$\Phi = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \Rightarrow \Phi^{-1} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} = e^{-At}$$

$$\vec{X} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \vec{C} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \underbrace{\int_0^t \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{-2s} \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} ds}_{\begin{pmatrix} -3e^{-s} \\ \frac{1}{2}e^{-2s} \end{pmatrix}\Big|_0^t}$$

$$\vec{X} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \vec{C} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} -3e^{-t} + 3 \\ \frac{1}{2}e^{-2t} - \frac{1}{2} \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} e^t \\ 0 \end{pmatrix} c_1 + \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix} c_2 + \begin{pmatrix} -3 + 3e^t \\ \frac{1}{2} - \frac{1}{2}e^{2t} \end{pmatrix}$$

redundant.

Relabel: $$\boxed{\vec{x} = c_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix}}$$

$c_3 = c_1 + 3$ $\qquad$ $c_4 = c_2 - \frac{1}{2}$

Sometimes if A isn't nice, we can make it nice:

Example: Solve the given system by diagonalizing the coefficient matrix.

$$\mathbf{X}' = \begin{pmatrix} 2 & 1 \\ -3 & 6 \end{pmatrix} \mathbf{X}$$

First find eigenvalues and eigenvectors.

$$\begin{vmatrix} 2-\lambda & 1 \\ -3 & 6-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 8\lambda + 15 = 0$$

$\lambda = 3, 5$

$\lambda_1 = 3$: $\begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix} \to \vec{K}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 5$: $\begin{pmatrix} -3 & 1 \\ -3 & 1 \end{pmatrix} \to \vec{K}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Form a matrix P with $\vec{K}_1$ & $\vec{K}_2$ as columns

So $P = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \Rightarrow P^{-1} = \dfrac{1}{-2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$

$$D = P^{-1}AP = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Note: $D^n = (P^{-1}AP)^n = \underbrace{(P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)}_{n \text{ times}}$

$$P D^n P^{-1} = P P^{-1} A^n P P^{-1} \Rightarrow A^n = P D^n P^{-1}$$

$$e^{At} = e^{PDtP^{-1}} = PP^{-1} + t\, PDP^{-1} + \frac{t^2}{2!} PD^2P^{-1} + \cdots$$

$$= P\left(I + tD + \frac{t^2}{2!}D^2 + \cdots\right)P^{-1}$$

Here $e^{At} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$

So $\boxed{\vec{X} = \begin{pmatrix} \frac{3}{2}e^{3t} - \frac{1}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{1}{2}e^{5t} \\ \frac{3}{2}e^{3t} - \frac{3}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{3}{2}e^{5t} \end{pmatrix}\vec{C}}$

eigenvalues $\Rightarrow$ eigenvectors $\Rightarrow P, P^{-1}, D$

$$e^{At}\vec{C} = P e^{Dt} P^{-1} \vec{C}$$